Linear and Piecewise Functions

Meaning of Linear Functions:
Linear functions describe situations where a value changes at a constant rate. That constant change is the slope of the function. The y-intercept of the function represents the initial amount (i.e. the answer when the input is zero). This indicates that linear functions can always be described using the pattern:

\[ y = mx + b. \]

Notice that there is no exponent written on either variable. This implies the exponent is “1” and that is what makes these equations’ graphs be straight lines. Thus, the name “linear” function. Mathematicians also use the word “degree 1” to describe linear functions.

For example, suppose a salesman earns a salary of $1000 per month AND a commission of $0.05 for every dollar’s worth of merchandise he sells. The $0.05 represents the constant change in the paycheck that varies with the amount of merchandise he sells. The initial amount is $1000 because this is the paycheck earned if he sells $0 of merchandise. In this situation the $0.05 is the slope and $1000 is the y-intercept. Thus using \( y = mx + b \) as our pattern, the equation that describes this salesman’s paycheck is:

\[ \text{pay amount} = \$0.05(\text{merchandise sold}) + \$1000. \]

The words “variable costs” and “steady change” also refer to the slope.
The words “fixed costs” and “original amount” refer to the y-intercept of the linear function.

Piecewise Functions:
Sometimes more than one set of criteria is needed to describe a situation. Here’s an example:
Below are the monthly rates for Cell Phones from four different companies. Our investigation will allow us to discover which company had the “best deal” for different types of customers.

<table>
<thead>
<tr>
<th>COMPANY</th>
<th>ACCESS FEE</th>
<th>MINUTES INCLUDED</th>
<th>$ per ADDITIONAL MINUTE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orange</td>
<td>$15.00</td>
<td>none</td>
<td>$0.70</td>
</tr>
<tr>
<td>Green</td>
<td>$25.00</td>
<td>150</td>
<td>$0.50</td>
</tr>
<tr>
<td>Blue</td>
<td>$50.00</td>
<td>600</td>
<td>$0.30</td>
</tr>
<tr>
<td>Red</td>
<td>$75.00</td>
<td>1500</td>
<td>$0.30</td>
</tr>
</tbody>
</table>

Answer these questions before continuing.

1. What is the input? (What will make the bill different from one month to another?)
2. What is the output? (What do we want to end up knowing?)
3. What is the fixed cost? (What will the customer have to pay regardless of usage?)
4. What is the variable cost? (What will change the monthly bill when usage is considered?)

Write the functions which calculate the monthly bills for customers of each company. Use the pattern:

\[ \text{output} = \text{variable cost (input)} + \text{fixed cost} \]

Again, try to do this before you look at the answers.

Orange:  
Green:  
Be careful here.
If \( \text{monthly\_bill} = \text{cost\_additional\_minute} \times (\text{minutes\_used}) + \text{access\_fee} \),
then what do we do with the minutes included in the plan?

Blue:  
Red:  

To analyze the plans for the “best buy” we can substitute values into each of our functions and calculate to see which has the smallest monthly bill, OR we can graph each function on the same graph. If we do that, we can analyze many situations quickly without repeatedly calculating through the same equations but with different input values.

Which company’s plan is the least expensive (best) buy for a customer who calls...

A. ...100 minutes per month?

B. ...300 minutes per month?

C. ...900 minutes per month?

D. ...1350 minutes per month?

E. What is the MOST expensive (worst) buy for 300 minutes per month?

How did you determine your answers?
ANSWERS:

1. The number of minutes used will affect the total of the monthly bill. The variable $x$ will represent minutes used.
2. We want to know the amount of the monthly bill. We will use $f(x)$ to represent this result.
3. The fixed cost is what the customer has to pay even if the phone is not used. This is the “access fee.” It will be the $y$-intercept of our equation.
4. The variable cost is affected by the charge per additional minute. That is our slope.

So far: \[
\text{monthly\_bill} = \text{cost\_additional\_minute} \times (\text{minutes\_used}) + \text{access\_fee}.
\]

Orange plan: \[\text{orange}(x) = 0.70(x) + 15\]

Green plan: \[
\text{green}(x) = \begin{cases} 
25.00 & \text{if } x < 150 \\
0.50(x - 150) + 25.00 & \text{if } x > 150
\end{cases}
\]

We use $x-150$ in the function so we don’t charge the customer for the minutes that are included in the plan.

- Since there are two cases to consider (under-usage and over-usage) the function must be defined in “pieces.” That’s why this is called a “piecewise” function.

Blue plan: \[
\text{blue}(x) = \begin{cases} 
50.00 & \text{if } x < 600 \\
0.30(x - 600) + 50.00 & \text{if } x > 600
\end{cases}
\]

Red plan: \[
\text{red}(x) = \begin{cases} 
75.00 & \text{if } x < 1500 \\
0.30(x - 1500) + 75.00 & \text{if } x > 1500
\end{cases}
\]

Which company’s plan is the least expensive (best) buy for a customer who calls...

A. ...100 minutes per month? Green
B. ...300 minutes per month? Blue
C. ...900 minutes per month? Red
D. ...1350 minutes per month? Red
E. What is the MOST expensive (worst) buy for 300 minutes per month? Orange

How did you determine your answers?

To find the best buy look for the color of the lowest line at that $x$-value.
To find the worst buy, look for the color of the highest line at that $x$-value.

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